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Extensions of the General Method of Generating Clifford Algebras and Comparisons with Various Existing Methods

Deming Li,[1] Charles P. Poole Jr.,[2] and Horacio A. Farach[2]

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In a continuation of previous work, we extend the general method of generating Clifford algebras based on a nonstandard intermediate step in the direct product procedure. This greatly simplifies the construction of the hierarchies of even- and odd-order Clifford algebras and facilitates comparison with other generating methods. Four other methods are compared. Various representations of Dirac matrices are derived in a unified way following our method.

1. INTRODUCTION

Since the pioneering work of Hestenes (1966; Hestenes and Sobczyk, 1984), the interest of physicists in Clifford algebras has increased significantly. Although a Clifford algebra (CA) has essentially only one inequivalent irreducible representation (Li *et al.*, 1986), for physical applications it is important to have a simple representation based, if possible, on the familiar Pauli-type matrices. Over the years many kinds of generating methods have been developed which differ widely from each other, and a general method that unifies the existing methods is lacking. In a previous article (Li *et al.*, 1986) we proposed a systematic general generating method (GM) for constructing higher order Clifford algebras from lower order ones. This method encompasses all possible binary generating methods and deals with universal Clifford algebras, so it also provides a systematic classification of universal Clifford algebras (Li *et al.*, 1986). In this article we present an extended version of GM, denoted EGM, which makes the construction even simpler and the connection with other methods more straightforward.

[1]Department of Physics, Shanxi University, Taiyuan, China.
[2]Department of Physics and Astronomy, University of South Carolina, Columbia, South Carolina 29208.

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The idea is to utilize a nonstandard "primitive generating set" (PGS) which differs from the standard one by the addition of just one element, the canonical element $\sigma(m)$, and the added element denoted by Σ_{m+1} can be either $\sigma(m)$ or $i\sigma(m)$. We will show that this makes the generating procedure more flexible and permits it to be condensed in a simple formula.

In Section 2 we begin by summarizing some important facts about Clifford algebras, meaning universal ones if not otherwise specified, and we fix the notation. Then in Section 3 we give a brief summary of our earlier GM, paving the way for presenting the extended generating method in Section 4, where a compact generating formula is derived. Section 5 applies the method to double field algebras. The relationships with our previous procedures (Li *et al.*, 1986) and with various other generating methods are discussed in Sections 6 and 7, respectively. Finally, representations of the Dirac algebra are treated in a unified way in Section 8 and some concluding remarks are made in Section 9. In a subsequent article we will give another extension to GM and compare it with other categories of generating methods.

2. UNIVERSAL CLIFFORD ALGEBRAS

A universal CA of order n is a real algebra on a linear vector space with n basis elements:

$$C_n(p, q): \quad \{\sigma_1, \sigma_2, \ldots, \sigma_p, \sigma'_{p+1}, \ldots, \sigma'_{p+q=n}\} \tag{1}$$

These elements anticommute with each other,

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 2g_{ij}1 \tag{2}$$

where g_{ij} is the metric tensor:

$$g_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \leq p \\ -1, & i = j > p \end{cases} \tag{3}$$

so, accordingly,

$$(\sigma_i)^2 = 1, \qquad i = 1, 2, \ldots, p \tag{4}$$

$$(\sigma'_j)^2 = -1, \qquad j = p+1, p+2, \ldots, p+q = n \tag{5}$$

Sometimes we use e instead of 1 for the unit element.

For our purpose it will be convenient to write A_n^s instead of $C_n(p, q)$, where the signature index s is defined by

$$s = p - q \tag{6}$$

For each order n, there are $n+1$ different signature indices differing by steps of two in the range

$$-n \leq s \leq n \tag{7}$$

and s is an even or odd integer depending on n being even or odd.

In Li *et al.* (1986) we showed that the properties of a particular CA depend critically upon the canonical element $\sigma(n)$, which is defined as the product of the basis elements

$$\sigma(n) = \sigma_1 \sigma_2 \cdots \sigma_p \sigma'_{p+1} \cdots \sigma'_{p+q} = n \tag{8}$$

The canonical element commutes with all of the basis elements when n is odd and anticommutes with all of them when n is even. It has a positive or negative square of unity depending upon the value of the signature index s in accordance with the following rules:

$$[\sigma(n)]^2 = \begin{cases} 1 & \text{when} \quad s = 0 \text{ or } 1 \pmod 4 \\ -1 & \text{when} \quad s = -1 \text{ or } 2 \pmod 4 \end{cases} \tag{9}$$

3. GENERAL GENERATING METHOD

In Li *et al.* (1986) we showed that the basis elements of a higher order CA A^w_{m+n} may be generated from a lower order one by forming direct products of the basis elements Σ_j of an even-order algebra A^s_m with those τ_k of another algebra A^t_n. We presented two general procedures (and there are only two!) for the systematic construction of the basis elements of A^w_{m+n}. Note, as indicated above, that for the even algebra A_m the canonical element $\sigma(m)$ anticommutes with all of the basis elements and has a square that equals e or $-e$, so it can be treated on the same footing as other basis elements in the construction procedure. Therefore, we introduce the notion of a "primitive generating set" (PGS) that comprises the following $m+1$ elements of the even-order algebra A_m:

$$\Sigma_k = \begin{cases} \sigma_i \text{ or } \sigma'_j & \text{when} \quad k = i, j \leq m \\ \sigma(m) & \text{when} \quad k = m+1 \end{cases} \tag{10}$$

Then the first procedure consists of forming direct product sets of

$$\begin{aligned} P[\Sigma_{k0}\times]: \quad & \{\Sigma_{k0} \times \tau_j\}, \quad j = 1, 2, \ldots, n \\ & \{\Sigma_k \times e_\tau\}, \quad k = 1, 2, \ldots, m+1, \quad k \neq k_0 \end{aligned} \tag{11}$$

where e_τ is the unit element of A_n and k_0 is fixed for the particular procedure, characterizing the basis set thus formed. The second procedure consists of forming the direct product set of

$$\begin{aligned} p[\times \tau_{j0}]: \quad & \{\Sigma_k \times \tau_{j0}\}, \qquad k = 1, 2, \ldots, m+1 \\ & \{e_\sigma \times \tau_j\}, \qquad j = 1, 2, \ldots, n, \quad j \neq j_0 \end{aligned} \tag{12}$$

where e_σ is the unit element of the algebra A_m. The final CA is clearly of order $m+n$. As for the signature index w, it is related to that of A_m^s and A_n^t as well as the characteristic element chosen for the procedure, namely Σ_{k0} or τ_{j0}. The various possibilities are listed in Tables I and II of Li *et al.* (1986).

Note that the reversed procedures such as

$$P[\times\Sigma_{k0}]: \quad \{\tau_i\times\Sigma_{k0};\ e_\tau\times\Sigma_k\} \tag{11a}$$

$$P[\tau_{j0}\times]: \quad \{\tau_{j0}\times\Sigma_k;\ \tau_j\times e_\sigma\} \tag{12a}$$

are not counted as new procedures, as these are equivalent to (11) and (12) in the sense that their differences may be reduced to the differences in the definitions of matrix direct product, either left product or right product. In applications both sets are widely used.

By selecting $m=2$ we obtained (Li *et al.*, 1986), a hierarchy of every kind of higher order CA.

4. EXTENDED GENERAL GENERATING METHOD

In the restricted generating method the last element σ_{m+1} of the primitive generating set from the algebra A_m^s is the canonical element $\sigma(m)$. The idea of the extended generating method EGM is that we drop this restriction, allowing Σ_{m+1} to be either $\sigma(m)$ or $i\sigma(m)$. Of course, in this way the corresponding "nonstandard" primitive generating set (NPGS) may develop a larger algebra A_{m+1} than the original one A_m, as when $\Sigma_{m+1}=i\sigma(m)$ and m is even. However, we do not need to worry about this, because what concerns us is the final algebra that is constructed. This being understood, the procedure of forming direct product basis elements becomes considerably simplified; it is no longer necessary to specify a procedure with a signature index s and the eight cases listed in Table I of Li *et al.* (1986) reduce to the four cases of Table I of the present paper.

Table I. Characteristics of the Four Procedures for Generating Higher Order Clifford Algebras by the Formation of Direct Products Using the Nonstandard Primitive Generating Set $\Sigma_{(m+1)}$ with Signature Index $s'=p'-q'$, and the Algebra A_n^t with Signature Index $t=u-v$

Procedure	Final signature (p, q)	Final signature index w $(=p-q)$
$P[\Sigma_{k0}]$	$(p'+u-1, q'+v)$	$s'+t-1$
$P[\Sigma'_{k0}\times]$	$(p'+v, q'+u-1)$	$s'-t+1$
$P[\times\tau_{j0}]$	$(p'+u-1, q'+v)$	$s'+t-1$
$P[\times\tau'_{j0}]$	$(q'+u, p'+v'-1)$	$-s'+t+1$

Note that we are dealing with nonstandard primitive generating sets $\Sigma_{(m+1)}$ of $m+1=\text{odd}$ elements having p' positive squared ones and q' negative squared ones. Here p' and q' are not the same as the p, q of the original algebra A_m^s from which $\Sigma_{(m+1)}$ is constructed. A_m is of even order, but the procedure (11) and (12) always gives

$$\Sigma_{(m+1)}^{s'} \times A_n^t = A_{m+n}^w \tag{13}$$

with $\Sigma_{(m+1)}: \{\Sigma_1, \Sigma_2, \ldots, \Sigma_m; \Sigma_{m+1}\}$, $\Sigma_{i<m+1} = \sigma_i$ or σ_j, or $\Sigma_{m+1} = \sigma(m)$ or $i\sigma(m)$. Of course, if we take $\Sigma_{m+1} = i\sigma(m)$, then $\Sigma_{(m+1)}$ would develop an $A_{m+1}^{s'}$ algebra, whereas if we take $\Sigma_{m+1} = \sigma(m)$, then $\Sigma_{(m+1)}$ would develop an A_m^s one. In either case equation (13) is obtained. As for the signature index s' of the primitive set $\Sigma_{(m+1)}$, it is defined in the same way as for a Clifford algebra: $s' = p' - q'$; however, it is a characteristic of $\Sigma_{(m+1)}^{s'}$, not of A_m^s.

As can be easily verified, the great simplification is due largely to the inclusion of Σ_{m+1} as an independent member when calculating $s' = p' - q'$ of $\Sigma_{(m+1)}$ (but not of A_m !). On the other hand, Table I is independent of Σ_{m+1} being $\sigma(m)$ or $i\sigma(m)$. Yet the inclusion of $i\sigma(m)$ as another alternative makes the procedure more flexible and the comparison with other constructing methods straightforward, as will be seen in the following sections.

As an example, we give in Table II the various NPGSs of $A_m^s\{\sigma_1, \sigma_2\}$ for the case $m=2$. Note that since the basis elements of a Clifford algebra are defined up to a sign $\pm$, we are free to choose $\pm\sigma(m)$ or $\pm i\sigma(m)$ for Σ_{m+1}, and in the table we select all Σ_3 to be positive.

Thus, to construct a hierarchy of Clifford algebras we need only one procedure $P[\Sigma_{k0}\times]$, and where $P[\Sigma_{(3)}^{s'} \times A_n^t] = A_{n+2}^{s'+t-1} = A_{n+2}^w$:

$$A_{n+2}^{s'+t-1} = \begin{cases} A_{n+2}^{t+2}: & s' = 3 \quad (L) \quad (14) \\ A_{n+2}^{t}: & s' = 1 \quad (M) \quad (15) \\ A_{n+2}^{t-2}: & s' = -1 \quad (R) \quad (16) \end{cases}$$

Figures 1 and 2 provide the hierarchies of even- and odd-order Clifford algebras, respectively. The letters L, M, and R, respectively, denote generating the algebra A_{n+2} from A_n^t by moving down to the left (L, $w = t+2$), directly down (M, $w = t$), or down to the right (R, $w = t-2$) on the figures.

Table II. The Nonprimitive Generating Sets $\Sigma_{(3)}^{s'}$ of the second-order algebra A_2^s

Algebra	NPGS	$(\Sigma_1, \Sigma_2; \Sigma_3)$	(p', q')	s'
A_2^2	$\Sigma_{(3)}^3$:	$(\sigma_1, \sigma_2; \sigma_3 = -i\sigma_1\sigma_2)$	(3, 0)	3
A_2^2	$\Sigma_{(3)}^1$:	$(\sigma_1, \sigma_2; \sigma_3' = \sigma_1\sigma_2)$	(2, 1)	1
A_2^0	$\Sigma_{(3)}^{-1}$:	$(\sigma_1, \sigma_2' = i\sigma_2; \sigma_3' = -i\sigma_1\sigma_2')$	(1, 2)	−1
A_2^{-2}	$\Sigma_{(3)}^{-3}$:	$(\sigma_1', \sigma_2'; \sigma_3' = -\sigma_1'\sigma_2')$	(0, 3)	−3

$$A_0^0$$

$$A_2^2 \quad A_2^0 \quad A_2^{-2}$$

$$A_4^4 \quad A_4^2 \quad A_4^0 \quad A_4^{-2} \quad A_4^{-4}$$

$$A_6^6 \quad A_6^4 \quad A_6^2 \quad A_6^0 \quad A_6^{-2} \quad A_6^{-4} \quad A_6^{-6}$$

Fig. 1. Hierarchy of even-order Clifford algebras formed by the extended generating procedure.

The position of an algebra in these hierarchies can be written down in a very compact form in terms of the left L, middle M, and right R operations. For the even-order case we have

$$A_{2n}^{w} = L^{w/2} M^{n-|w|/2} A_0^0, \qquad w \geq 0 \tag{17a}$$

$$= R^{-w/2} M^{n-|w|/2} A_0^0, \qquad w \leq 0 \tag{17b}$$

$$A_1^1 \quad A_1^{-1}$$

$$A_3^3 \quad A_3^1 \quad A_3^{-1} \quad A_3^{-3}$$

$$A_5^5 \quad A_5^3 \quad A_5^1 \quad A_5^{-1} \quad A_5^{-3} \quad A_5^{-5}$$

$$A_7^7 \quad A_7^5 \quad A_7^3 \quad A_7^1 \quad A_7^{-1} \quad A_7^{-3} \quad A_7^{-5} \quad A_7^{-7}$$

Fig. 2. Hierarchy of odd-order Clifford algebras formed by the extended generating procedure.

where $L^{w/2} = R^{-w/2}$ and in actuality both equations are valid for the entire range of values of w, namely $-n \leq w \leq n$. For odd order n it is convenient to write

$$A^w_{2n+1} = L^{(w-1)/2} M^{n-|w-1|/2} A^1_1, \qquad w \geq 1 \tag{18a}$$

$$A^w_{2n+1} = R^{-(w+1)/2} M^{n-|w+1|/2} A^{-1}_1, \quad w \leq -1 \tag{18b}$$

but this time the two equations are valid only for their own range of values of w. Besides, they differ in their starting algebras. The operators L, M, and R, of course, all commute with each other. As examples, consider

$$A^4_6 = L^2 M A^0_0$$

$$A^{-3}_7 = R M^2 A^{-1}_1$$

Expressions (17) and (18) locate a particular algebra A^w_m in the hierarchy by the number of moves to the left and down (L), directly down (M), and to the right and down (R) from the position of the starting algebra A^0_0 or $A^{\pm 1}_1$.

Now expressions (17) and (18) can be put into one compact expression if we introduce the "modified Gauss symbol"

$$[x]' \equiv \frac{x}{|x|}[|x|] \tag{19}$$

where $[x]$, the Gauss symbol, means the largest integer part of X. Thus, for example,

$$[\pi]' = [\pi] = 3$$

$$[-\tfrac{5}{2}]' = (-1)[\tfrac{5}{2}] = -2, \qquad \text{whereas} \quad [-\tfrac{5}{2}] = -3$$

Then, converting all R operators to L operators through the expression $R^x = L^{-x}$, we arrive at a single compact formula in place of expressions (17) and (18):

$$A^w_m = L^{[w/2]'} M^{[m/2]-|[w/2]'|} A^{S(w)}_{|S(w)|} \tag{20a}$$

or for characterization simply

$$A^w_m =: \{[w/2]', [m/2] - |[w/2]'|; S(w)\} \tag{20b}$$

where

$$S(w) = \begin{cases} 0, & w \text{ even} \\ 1, & w \text{ positive odd} \\ -1, & w \text{ negative odd} \end{cases}$$

			R_0^0			
		R_2^2	R_2^0	H_2^{-2}		
	H_4^4	R_4^2	R_4^0	H_4^{-2}	H_4^{-4}	
H_6^6	H_6^4	R_6^2	R_6^0	H_6^{-2}	H_6^{-4}	R_6^{-6}

Fig. 3. Hierarchy of even-order Clifford algebras formed by the extended generating procedure and written in a notation that distinguishes the real field (R) from the Hamilton quaternion (H) types.

5. PRIMITIVE GENERATING SETS

The extended generating method uses a direct product combination of a nonprimitive set $\Sigma_{(m+1)}^{s'}$ of basis type elements of the Clifford algebra A_n^t to provide a basis for the generated algebra A_{m+n}^w. Under certain conditions this generated set of $m+n$ elements will constitute the true basis for algebra A_{m+n}^w, and in this case it is referred to as a primitive basis set. Under other conditions the generated elements must be modified to convert them to true basis elements, and when this occurs the set is called nonprimitive. Whether the generated set is primitive or not depends upon the type of algebra that is formed, and so, before preceding, we will say a few words about the various types of algebras and how they fit into the hierarchies displayed in Figures 1 and 2.

We present in Figures 3 and 4, respectively, the Clifford algebra hierarchies of even order and odd order written in a notation that designates their carrier fields (Li *et al.*, 1986; Porteous, 1969; Poole and Farach, 1982), namely the real numbers (R), the complex numbers (C), and Hamilton's quaternions (H). The even-order algebras are based on the real number and quaternion fields and half of the odd-order algebras are based on the complex number field C. The remaining odd-order algebras are of the double-field type ${}^2R$ or ${}^2H$ based on the real number and quaternion fields,

			${}^2R_1^1$	C_1^{-1}			
		C_3^3	${}^2R_3^1$	C_3^{-1}	${}^2H_3^{-3}$		
	${}^2H_5^5$	C_5^3	${}^2R_5^1$	C_5^{-1}	${}^2H_5^{-3}$	C_5^{-5}	
C_7^7	${}^2H_7^5$	C_7^3	${}^2R_7^1$	C_7^{-1}	${}^2H_7^{-3}$	C_7^{-5}	${}^2R_7^{-7}$

Fig. 4. Hierarchy of odd-order Clifford algebras formed by the extended generating procedure. There is a complex number field type (C) and two double-field real (${}^2R$) and quaternion (${}^2H$) varieties.

respectively. We will now examine how the primitivities of the generated basis elements are related to the algebra types in these hierarchies.

The inequivalent irreducible representation of a Clifford algebra A_n^t of even order n is a set of $2^{n/2}\times 2^{n/2}$ square matrices (Li *et al.*, 1986; Porteous, 1969; Poole and Farach, 1982). The operations L, M, and R generate the algebras A_{n+2}^2 from A_n^t with $w=t$, $t\pm 2$ by forming a new set of w matrices of size $s^{(n+2)/2}\times 2^{(n+2)/2}$. All of the involved algebras are of type R or H, so the set of $n+2$ matrices that is formed is primitive and hence constitutes a true basis without any modifications or additional operations. Thus, there are no complicating factors in this procedure for generating even-order algebras.

The situation is more complicated with odd-order algebras. When the generated algebra A_{n+2}^w is a complex number field type C_{n+2}^w then the basis set which is generated is a true basis for the algebra and we call it a primitive basis. This is not the case with a generated double-field algebra ${}^2R_{n+2}^w$ or ${}^2H_{n+2}^w$ for which the basis is nonprimitive. Such a nonprimitive basis forms an algebra in which each element of the next lower-order even algebra R_{n+1}^s or H_{n+1}^s appears twice, hence the name double field. For example, the double-field algebra $A_3^{-3}={}^2H_3^{-3}$ has the three primed Pauli matrices $\sigma_j'=i\sigma_j$ as its nonprimitive basis. These generate an algebra with the following eight elements:

Unit element	I	
Basis elements	$\sigma_1', \sigma_2', \sigma_3'$	(21)
Binary products	$\sigma_3'\sigma_2'=\sigma_1'$; $\quad\sigma_1'\sigma_3'=\sigma_2'$; $\quad\sigma_2'\sigma_1'=\sigma_3'$	
Ternary product	$\sigma_1'\sigma_2'\sigma_3'=\sigma(3)=I$	

We see that each element of this double-field algebra appears twice.

A primitive basis Σ_j' for this double-field algebra ${}^2H_3^{-3}$ can be formed from the following reducible (already reduced) representation (Porteous, 1969):

$$\Sigma_1'=\begin{pmatrix}\sigma_1 & 0\\ 0 & -\sigma_1'\end{pmatrix},\qquad \Sigma_2^1=\begin{pmatrix}\sigma_2 & 0\\ 0 & -\sigma_2'\end{pmatrix},\qquad \Sigma_3^1=\begin{pmatrix}\sigma_3' & 0\\ 0 & -\sigma_3'\end{pmatrix}\tag{22}$$

which has the three binary products $\Sigma_i\Sigma_j$,

$$\Sigma_3'\Sigma_2'=\begin{pmatrix}\sigma_1' & 0\\ 0 & \sigma_1'\end{pmatrix},\qquad \Sigma_1'\Sigma_3'=\begin{pmatrix}\sigma_2' & 0\\ 0 & \sigma_2'\end{pmatrix},\qquad \Sigma_2'\Sigma_1'\Sigma_1'=\begin{pmatrix}\sigma_3' & 0\\ 0 & \sigma_3'\end{pmatrix}\tag{23}$$

the ternary product or canonical element,

$$\Sigma_1'\Sigma_2'\Sigma_3'=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}\tag{24}$$

and the unit element $\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$. All eight elements of this representation are different and all eight are in the reduced or diagonal form which is characteristic of a double-field algebra. In like manner, the double-field algebra ${}^2R_3'$ with the nonprimitive basis σ_1, σ_2, σ_3' can be generated from the following primitive basis set:

$$\Sigma_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \qquad \Sigma_2 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \qquad \Sigma_3' = \begin{pmatrix} \sigma_3' & 0 \\ 0 & -\sigma_3' \end{pmatrix} \tag{25}$$

using the same procedure.

We infer that a general odd-order double-field algebra of either the ${}^2R_m^w$ or ${}^2H_m^w$ type which has the m nonprimitive basis elements τ_j has a corresponding primitive basis set of the type

$$\Sigma_j = \begin{pmatrix} \tau_j & 0 \\ 0 & -\tau_j \end{pmatrix}, \qquad j = 1, 2, \ldots, m \tag{26}$$

When our extended generating method is employed to form such a double-field algebra, then the resulting nonprimitive basis set τ_j can always be converted to a primitive basis via the prescription of equation (26). For example, the Dirac algebra ${}^2H_5^5$ with the nonprimitive basis set

$$\begin{gathered} \gamma_1 = \begin{pmatrix} 0 & \sigma_1' \\ -\sigma_1' & 0 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 0 & \sigma_2' \\ -\sigma_2' & 0 \end{pmatrix}, \qquad \gamma_3 = \begin{pmatrix} 0 & \sigma_3' \\ -\sigma_3' & 0 \end{pmatrix} \\ \gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \end{gathered} \tag{27}$$

formed from the primed Pauli matrices σ_j' has the following five primitive basis elements Σ_j:

$$\Sigma_j = \begin{pmatrix} \gamma_j & 0 \\ 0 & -\gamma_j \end{pmatrix}, \qquad j = 1, 2, 3, 4, 5 \tag{28}$$

Dirac-type algebras of even order A_4^w will be discussed below in Section 7.

6. COMPARISON WITH PREVIOUS GENERATING METHODS

The general generating method for Clifford algebras that was developed in Li *et al.* (1986) defined the left (L'), middle (M'), and right (R') generating operators as follows:

$$\begin{aligned} L'A_n^t &= A_{n+2}^{t+4} \\ M'A_n^t &= A_{n+2}^t \\ R'A_n^t &= A_{n+2}^{t-4} \end{aligned} \tag{29}$$

[in Li *et al.* (1986) the prime notation was not used with L, M, and R]. These operations transform an A_n^t algebra into an A_{n+2}^w algebra where $w=t$, $t\pm 4$. This means skipping a column in the hierarchies of Figures 1 and 2. In the present work the analogous left (L), middle (M), and right (R) generating operators act as follows:

$$\begin{aligned} LA_n^t &= A_{n+2}^{t+2} \\ MA_n^t &= A_{n+2}^t \\ RA_n^t &= A_{n+2}^{t-2} \end{aligned} \tag{30}$$

corresponding to the generation of an $(n+2)$th-order algebra A_{n+2}^w of signature index $w=t$, $t\pm 2$ from the nth-order algebra A_n^t. Note that the two middle operators are identical ($M=M'$). These L, M, R EGM operators permit all of the even- and almost all of the odd-order algebras to be generated from either A_1^1 or A_1^{-1}. We say "almost all" because the A_w^w algebra cannot be so generated from A_1^{-1}, and the A_w^{-w} ones are inaccessible to A_1^1. Thus, it is advantageous to use L, R instead of the earlier operators L', R', which skip every other column in the generating process.

The difference between the previous L', R' and the present L, R generating operators is particularly striking in the odd-order case. We see from Figure 4 that the earlier operators L' and R' converted C-type algebras to higher order C-type ones, and double-field algebras to higher order double-field ones. More specifically, both L' and R' convert C_n^t to $C_{n+2}^{t\pm 4}$, ${}^2R_n^t$ to ${}^2H_{n+2}^{t\pm 4}$, and H_n^t to $R_{n+2}^{t\pm 4}$. The present left L and right R operators convert a complex number algebra C_n^t to a double-field type ${}^2R_{n+2}^{t\pm 2}$ or ${}^2H_{n+2}^{t\pm 2}$ and the double-field algebras ${}^2R_n^t$ and ${}^2H_n^t$ to complex number ones $C_{n+2}^{t\pm 2}$. The significance of this was explained in the previous section.

7. COMPARISON WITH OTHER GENERATING METHODS

(1) The method of Brauer and Wey (1935; see also Boerner, 1955, Chapter VIII, Section 3; Carson, 1953, p. 164).

(a) Even-order case A_{2n}^{2n}: From (18) we see that $A_{2n}^{2n}=L^nA_0^0$ and from Table II we have $\Sigma_{(3)}^3=\{\sigma_1,\sigma_2,\sigma_3\}$, where σ_1, σ_2, σ_3 are the Pauli matrices. For the procedure $P[\Sigma_{k0}\times]$ to generate the algebra $A_{m+n}^w=A_4^4$ from $A_m^s=A_2^2$ and $A_n^t=A_2^2$, we take Σ_{k0} to be $\sigma_3=-\sigma(2)=-i\sigma_1\sigma_2=\Sigma_{2+1}$, the original A_2^2 being $A_2^2\{\sigma_1,\sigma_2\}$. Then from (11) we have $\tau_1=\sigma_1$ and $\tau_2=\sigma_2$, and the basis elements of A_4^4 are

$$\begin{aligned} \{\Sigma_{k0}\times\tau_j\} &\to \sigma_3\times\sigma_1 \qquad \sigma_3\times\sigma_2 \\ \{\Sigma_k\times e_\tau\} &\to \sigma_1\times e \qquad \sigma_2\times e \end{aligned} \tag{31}$$

These four elements are the basis elements of A_4^4 and will be treated as τ_j elements in the next step of construction. Starting from A_4^4, by applying one more L procedure we generate the algebra $A_{m+n}^w = A_6^6$ using the same $\Sigma_{k0} = \sigma_3$. Note that this time for e_τ we have $e_\tau = e \times e$; therefore

$$\begin{array}{ll} \{\Sigma_{k0} \times \tau_j\} \to \sigma_3 \times \sigma_3 \times \sigma_1 & \sigma_3\sigma_3 \times \sigma_3 \\ \sigma_3 \times \sigma_1 \times e & \sigma_3 \times \sigma_2 \times e \\ \{\Sigma_k \times e_\tau\} \to \sigma_1 \times e \times e & \sigma_2 \times e \times e \end{array}$$

Further construction of the algebra A_8^8 from A_6^6 using the same nonstandard primitive generating set Σ_3 clearly preserves the same pattern:

$$\begin{array}{ll} \sigma_3 \times \sigma_3 \times \sigma_3 \times \sigma_1 & \sigma_3 \times \sigma_3 \times \sigma_3 \times \sigma_2 \\ \sigma_3 \times \sigma_3 \times \sigma_1 \times e & \sigma_3 \times \sigma_3 \times \sigma_2 \times e \\ \sigma_3 \times \sigma_1 \times e \times e & \sigma_3 \times \sigma_2 \times e \times e \\ \sigma_1 \times e \times e \times e & \sigma_2 \times e \times e \times e \end{array}$$

and we have finally two sets of n basis elements each for the algebra A_{2n}^{2n}:

$$\begin{aligned} \alpha_j&: \quad \{\sigma_3 \times \sigma_3 \times \cdots \times \sigma_3 \times \sigma_1 \times e \times \cdots \times e\} \\ \beta_j&: \quad \{\sigma_3 \times \sigma_3 \times \cdots \times \sigma_3 \times \sigma_2 \times e \times \cdots \times e\} \end{aligned} \tag{32}$$

(b) Odd-order case A_{2n+1}^{2n+1}: Since we are dealing with a real algebra, the two sets α_j, β_j and $i\alpha_j$, $i\beta_j$ are independent. The basis for A_{2n+1}^{2n+1} is that of A_{2n}^{2n}, namely the α_j, β_j of (32) plus Σ_{2n+1}, which is i times the canonical element $\sigma(2n)$. In other words, $\Sigma_{2n+1} = i\sigma(2n)$, so we have

$$\begin{aligned} \Sigma_{2n+1} &= i\alpha_1\alpha_2 \cdots \alpha_n\beta_1\beta_2 \cdots \beta_n \\ &= i(-1)^{(n-1)^2/2}\alpha_1\beta_1\alpha_2\beta_2 \cdots \alpha_n\beta_n \\ &= i(-1)^{(n-1)^2/2}(-1)^n\sigma_3 \times \sigma_3 \times \cdots \times \sigma_3 \\ &= (-1)^{(n^2+1)/2} i^{n+1}\sigma_3 \times \sigma_3 \times \cdots \times \sigma_3 \\ &= \sigma_3 \times \sigma_3 \times \cdots \times \sigma_3 \end{aligned}$$

irrespective of whether n is even or odd, where a possible unimportant ± 1 sign is neglected. Now, clearly the NSPGS $\{\Sigma_1, \Sigma_2, \ldots, \Sigma_{2n}; \Sigma_{2n+1}\}$ develops the algebra A_{2n+1}^{2n+1}. This is essentially the same as in Brauer and Weyl (1935).

(2) The method of Srivastava (1982) for even algebras. There are two cases to consider. For $m = 0 \pmod 4$, we construct $A_{m+2}^{m+2} = A_2^2 \times A_m^m$ through the procedure $P[\times\tau_{j0}]$, and for a representation of A_2^2 we take the real algebra $A_2^2\{\sigma_1, \sigma_3\}$. Hence

$$\Sigma_k \times \sigma_3; \quad 1 \times \sigma_1 \tag{33}$$

where for the basis elements of A_m^m we follow Srivastava and use β_μ, so $\Sigma_k \equiv \beta_k$ $(k=m)$; $\Sigma_{m+1} \equiv \sigma(m) = \beta_1\beta_2 \cdots \beta_m$, and $(\Sigma_{m+1})^2 = 1$ [cf. Li *et al.*, 1986, equation (23)].

For $m=2 \pmod 4$ we have $[\sigma(m)]^2 = -e$, and to form a restricted CA with $\sigma_i^2 = 1$ for the entire basis we must choose $\Sigma_{m+1} \equiv i\sigma(m) = i\beta_1\beta_2 \cdots \beta_m$ and then proceed as above.

For comparison with Srivastava's results, we must make use of the left-hand direct product definition

$$A \times B \equiv \begin{array}{c} AB_{11} \quad AB_{12} \cdots AB_{1n} \\ \text{------------------} \\ AB_{n1} \text{----------} AB_{nn} \end{array} \tag{34}$$

and equation (33) assumes the form

$$A_{m+2}^{m+2}: \quad \left\{ \begin{bmatrix} \beta_\mu & 0 \\ 0 & -\beta_\mu \end{bmatrix}; \begin{bmatrix} \Sigma_{m+1} & 0 \\ 0 & -\Sigma_{m+1} \end{bmatrix}; \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} \tag{35}$$

which is identical with that in Srivastava (1982).

(3) The method of Ramakrishnan (1970). This is essentially our extended procedure

$$\Sigma_{(3)} \times A_n \to A_{n+2}$$

with $\Sigma_{(3)}: \{\sigma_1, \sigma_2, \sigma_3\}$. Applying procedure (11), choosing Σ_{k0} to be σ_1, we have

$$\{\sigma_1 \times \tau_j^{(n)}, \sigma_2 \times 1, \sigma_3 \times 1\} \tag{36}$$

Actually, this is enough, but Ramakrishnan gives (36) only through its linear combination:

$$\hat{L}_{(n)} \equiv \sum_k \lambda_k \tau_k^{(n)} \tag{37}$$

These results mean that we have a linear combination of order $n+2$:

$$\begin{aligned} \hat{L}_{(n+2)} &= \sigma_1 \times \hat{L}_{(n)} + \lambda_{n+1}\sigma_2 \times 1 + \lambda_{n+2}\sigma_3 \times 1 \\ &= \begin{bmatrix} \lambda_{n+2}1 & L(n) - i\lambda_{n+1}1 \\ L(n) + i\lambda_{n+1}1 & -\lambda_{n+2}1 \end{bmatrix} \end{aligned} \tag{38}$$

where the "right direct product" definition is being followed. Once equation (38) is formed from $\hat{L}(n)$, then (36) is uniquely determined, as was explained by Ramakrishnan (1970).

(4) The method of our previous work (Poole and Farach, 1982).

Let the PGS be $\Sigma_{(3)}: \sigma_3, \sigma_1; \sigma_2'\}$, $s=1$, and take $\Sigma_{k0}' \equiv \sigma_2' = i\sigma_2$; then procedure (11) gives

$$\{\sigma_2' \times \tau_j; \sigma_3 \times 1, \sigma_1 \times 1\} \tag{39}$$

In matrix form, using the right direct product, we have

$$A_{n+2}^{-t+2}: \quad \tau_{k\le n}^{(n+2)} = \begin{bmatrix} 0 & \tau_k^{(n)} \\ -\tau_k(n) & 0 \end{bmatrix}; \quad \tau_{n+1}^{(n+2)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \quad \tau_{n+2}^{(n+2)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdots \tag{40}$$

where t is the index of A_n; see Table I for $P[\Sigma'_{k0}\times]$. According to Poole and Farach (1982), we first keep n odd and construct all odd-order CAs. Then, simply by dropping the last basis element $\tau_{n+2}^{(n+2)}$, we arrive at the corresponding lower-order even algebras. This gives Table III of Poole and Farach (1982). In this way we obtain $[\tau_{k\le n}^{(n+2)}]^2 = -[\tau^{(n)}]^2$ from the table of Poole and Farach (1982). The advantage of this approach is that we have exclusively real representations. If we want to stick to the restricted CA, we must use σ_2 instead of σ'_2 as Σ_{k0}.

These few examples examined above suggest that many more varieties of generated algebra forms could be obtained just by applying different procedures, different choices of characteristic elements, and different versions of the direct product. Our comparisons demonstrate their equivalence to our method.

8. THE DIRAC MATRICES AND EDDINGTON'S *E* NUMBERS

Because of their importance in physics, the Dirac matrices deserve special treatment, and we will show that every well-known type of representation is just a variant of the six versions which follow directly from our EGM.

Of course, we are here dealing with equivalent representations, as there is only one inequivalent irreducible representation of the Dirac algebra (Li *et al.*, 1986). We choose representations which are convenient for physics, namely those which use Pauli matrices as building blocks and treat the three Dirac matrices γ_1, γ_2, γ_3 on the same footing.

Upon inspecting Table I and keeping in mind that the three Pauli matrices are to be treated on the same footing, it is evident that we must follow the procedure

$$P[\Sigma_{(3)}^{s'} \times \tau'_{j0}] \tag{41}$$

with $s' = -3$ and, from Tables I and II, the signature index $w = t-2$. Now the Dirac algebra belongs to the H-type algebra A_4^{-2}, since for the basis $\gamma_0^2 = 1$, $\gamma_k^2 = -1$ for $k = 1, 2, 3$, and hence $t = 0$, so there are only six possible choices of basis elements for A_2^0 in terms of Pauli matrices $\Sigma_{1,2,3}$ and $\sigma'_{1,2,3}$. This makes a total of six equivalent representations of γ_μ that are generated.

They are as follows:

(a) A_2^0: $\{\sigma_1', \sigma_2\}$: $\Sigma_{(3)}^3 = \{\sigma_1, \sigma_2, \sigma_3\}$: $\tau_{j0}' \equiv \sigma_1'$.

From (41), it follows that A_4^{-2}: $\{\sigma_k \times \sigma_1', 1 \times \sigma_2\}$ is generated with the following matrix forms, using the left direct product of equation (34):

$$\gamma_0 = \begin{bmatrix} 0 & -i1 \\ i1 & 0 \end{bmatrix}; \qquad \gamma_k = \begin{bmatrix} 0 & i\sigma_k \\ i\sigma_k & 0 \end{bmatrix}, \quad k = 1, 2, 3 \tag{42}$$

The three matrices $-i\gamma_k$ are often called the α_k matrices (Poole and Farach, 1982; Arfken, 1985; Schiff, 1949; Schweber, 1962, pp. 69, 79).

(b) A_2^0: $\{\sigma_2', \sigma_1\}$, $\Sigma_{(3)}^3 = \{\sigma_1, \sigma_2, \sigma_3\}$, $\tau_{j0}' = \sigma_2'$.

Similarly, using the left direct product, we obtain

$$A_4^{-2}: \quad \{\sigma_k \times \sigma_2'; 1 \times \sigma_1\}$$

$$\gamma_o = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \qquad \gamma_k = \begin{bmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{bmatrix}, \quad k = 1, 2, 3 \tag{43}$$

This is the Weyl representation used, for example, by Schweber (1962).

(c) A_2^0: $\{\sigma_2', \sigma_3\}$: Likewise, we have

$$A_4^{-2}: \quad \{\sigma_k \times \sigma_2', 1 \times \sigma_3\}$$

$$\gamma_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \qquad \begin{bmatrix} 0 & \sigma \\ -\sigma_k & 0^k \end{bmatrix}, \quad k = 1, 2, 3 \tag{44}$$

the matrix γ_0 is sometimes called β (Arfken, 1985; Schiff, 1949; Schweber, 1962). This is the Pauli representation mentioned, for example, by Roman (1969, p. 615).

(d) A_2^0: $\{\sigma_3', \sigma_3\}$:

$$A_4^{-2}: \quad \{\sigma_k \times \sigma_3'; 1 \times \sigma_2\}$$

$$\gamma_0 = \begin{bmatrix} 0 & -i1 \\ i1 & 0 \end{bmatrix}; \qquad \gamma_k = \begin{bmatrix} i\sigma_k & 0 \\ 0 & -i\sigma \end{bmatrix}, \quad k = 1, 2, 3 \tag{45}$$

These γ_k matrices are sometimes called $i\delta_k$ (Poole and Farach, 1982; Arfken, 1985).

(e) A_2^0: $\{\sigma_3', \sigma_1\}$;

$$A_4^{-2}: \quad \{\sigma_k \times \sigma_3', 1 \times \sigma_1\}$$

$$\gamma_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \qquad \gamma_k = \begin{bmatrix} i\sigma_k & 0 \\ 0 & -i\sigma_k \end{bmatrix}, \quad k = 1, 2, 3 \tag{46}$$

We again have $\delta_k = -i\gamma_k$ for the basis (Poole and Farach, 1982; Arfken, 1985).

(f) $A_2^0: \{\sigma_1', \sigma_3\}$:

$$A_4^{-2}: \quad \{\sigma_k \times \sigma_1', 1 \times \sigma_3\}$$

$$\gamma_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \qquad \gamma_k = \begin{bmatrix} 0 & i\sigma_k \\ i\sigma_k & 0 \end{bmatrix}, \quad k = 1, 2, 3 \tag{47}$$

Again we have $\beta = \gamma_0$ (Ramakrishnan, 1970; Arfken, 1985; Schiff, 1949) and $\alpha_k = i\gamma_k$ (Srivastava, 1982; Ramakrishnan, 1970; Arfken, 1985; Schiff, 1949). These expressions (42–47) are the popularly used sets of the Dirac-type matrices (Arfken, 1985).

There are other CAs equivalent to the Dirac algebra. An interesting example is Eddington's E numbers (Roman, 1969; Eddington, 1946), which can be given in terms of Pauli matrices σ_i as follows (Eddington, 1946):

$$E_1 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix},$$

$$E_4 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}, \qquad (E_\mu)^2 = -1, \mu = 1, 2, 3, 4 \tag{48}$$

This Eddington algebra is A_4^{-4} and comparing Figures 1 and 3, we see that it is equivalent to A_4^{-2}, both being of type H. The E numbers are constructed through our procedure starting from

$$A_2^2: \quad \{\sigma_2, \sigma_3\} \qquad \text{and} \qquad \Sigma_{(3)}^{-3}: \quad \{i\sigma_2, i\sigma_1, i\sigma_3\}$$

and choosing Σ'_{k0} to be $i\sigma_2$; then it follows from (11) that

$$P[\Sigma'_{k0}\times]: \quad \left\{ i\sigma_2 \times \begin{Bmatrix} \sigma_2 \\ \sigma_3 \end{Bmatrix}; \begin{Bmatrix} i\sigma_1 \\ i\sigma_3 \end{Bmatrix} \times 1 \right\} \tag{49}$$

Written in matrix form, (49) is just (48) provided the left-hand direct product is understood.

Finally, for the interesting Majorana representation of the H-type algebra A_4^{-2} we have to follow the $P[\Sigma_0\times]$ procedure with

$$\Sigma_{(3)}^{-2} = \{\sigma_1', \sigma_2, \sigma_3'\}, \qquad \Sigma_{k0} = \sigma_2, \qquad \text{and} \qquad A_2^0: \quad \{\sigma_1, \sigma_2'\}$$

Thus

$$P[\Sigma_{k0}\times] = \{\sigma_2 \times \sigma_1, \sigma_2 \times \sigma_2'; \sigma_3' \times 1, \sigma_1' \times 1\} \tag{50}$$

If the left-hand direct product is used, these assume the same form as that given in Bacry (1977, p. 402), except for the unimportant sign of ± 1. Because of its pure imaginary nature, the Majorana representation is important for discussions connected with charge-conjugation transformations.

The Majorana (1937) algebra, however, is different from the Dirac algebra. In our notation it is A_4^2. To obtain its matrix representation, we take the NPGS $\Sigma_{(3)}^3=\{\sigma_1, \sigma_2, \sigma_3\}$ as the generating set and A_2^0: $\{\sigma_1', \sigma_2\}$ as the starting algebra. Then the application of procedure $P[\Sigma_{k0}\times]$ gives, according to Table I, an index of $w=3+0-1=2$. Specifically, if we choose $\Sigma_{k0}=\sigma_2$, then procedure (11) gives

$$A_4^2\colon\quad \left\{\sigma_2\times\begin{Bmatrix}\sigma_1'\\ 2\end{Bmatrix};\quad \begin{Bmatrix}\sigma_1\\ \sigma_3\end{Bmatrix}\times 1\right\} \tag{51}$$

For comparison, we apply the left-hand direct product (34), and (51) assumes the following form:

$$A_4^2\colon\quad \begin{pmatrix}0 & \sigma_2'\\ \sigma_2 & 0\end{pmatrix},\ \begin{pmatrix}0 & -\sigma_2'\\ \sigma_2' & 0\end{pmatrix},\ \begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_1\end{pmatrix},\ \begin{pmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{pmatrix} \tag{52}$$

which, except for an unimportant difference of sign, is the same as that given in Table I of Salingaros and Dresden (1983).

9. CONCLUSION

In this article we have developed a systematic procedure for generating a Clifford algebra of an arbitrary order and any desired signature index. We believe that the potentiality and flexibility of the new method are quite clear, but its most important advantage is that it provides a systematic approach for constructing a representation of any order and index with the required properties, such as unitarity and Hermiticity. This advantage was illustrated by presenting the procedures which correspond to several generating methods proposed by others, and by forming the Eddington E-number matrices, the Majorana algebra, and various versions of the Dirac-type matrices which have been widely used in theoretical physics.

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